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A parameter perturbation technique applied to multipoint iteration functions for the solution of systems of nonlinear equations

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A PARAMETER PERTURBATION TECHNIQUE APPLIED TO MULTIPOINT ITERATION FUNCTIONS FOR THE SOLUTION OF SYSTEMS OF NONLINEAR EQUATIONS

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ABSTRACT

The convergence of classical iterative procedures, when applied to a system of nonlinear algebraic or transcendental equations, is highly dependent upon a good initial approximation to the desired roots.

Most of the classical iterative schemes have convergence factors between one and two. In this paper iterative schemes of order two and greater are studied in connection with a parameter perturbation process. The parameter perturbation process relaxes the restrictions on the choice of initial values. The procedure divides each problem into a number of subsidiary problems. Each subsidiary system of equations is then solved until a solution is found to the original problem.

The study presents a discussion of the iteration functions chosen, of the parameter perturbation algorithm and the conditions for convergence.

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I. INTRODUCTION

In the numerical solution of a system of nonlinear equations, the convergence of classical procedures is highly dependent upon a good initial approximation to the desired roots. The restriction of a good initial approximation to the solution, has greatly limited the use of these methods.

New sets of iteration functions have been introduced by J. F. Traub(1)*. These new classes of iteration functions have convergence factors (orders) of two or greater and should be more effective in relaxing the restriction on the choice of starting values.

Ferdinand Freudenstein and Bernard Roth(2) discuss a procedure they developed while working with Path-Generating Mechanisms(3), which further relaxes the restriction on the choice of starting values. The procedure is one of parameter perturbation, in which the parameters of a derived system, with a known solution, are successively incremented a finite number of times until they are equal to the parameters of the desired system. The various systems formed by successively incrementing the parameters of the desired system belong to the same family as the unknown system and differ only in the parameters.

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*All numbers (x) refer to the bibliography while the numbers (x.y) refer to equations.

The solution of the jth member of this set is used as the starting value for the iterative solution of the (j + 1)'st member of the set until a solution is found to the desired unknown system. In constrast to standard numerical procedures, not only is the trial solution altered, but, also, the constants of the equations themselves.

The purpose of this paper is to determine the utility of the new sets of iteration functions in connection with the parameter perturbation procedure.

The functional equations used are first developed for the case of one equation and one unknown and are then generalized to systems.

II. REVIEW OF LITERATURE

An examination of available literature reveals that there are in general no direct methods for solving systems of nonlinear equations. Work on the solution of systems of nonlinear equations has proceeded in many directions. Hooke and Jeeves(4) have developed a search technique in n dimensions which may prove very useful. G. C. Caldwell(5), W. C. Davidon(6) and J. A. Ward(7) have developed methods of minimizing a system of nonlinear equations which may also be used to find the solution to a system of nonlinear equations. A. N. Gleyzal(8) describes a procedure that involves minimizing the sum of the squares for each function in a system. These methods were not investigated as they fall outside the scope of this thesis.

The solution of systems of linear equations was not investigated as this subject has a vast literature of its own.

The problem is to find a root, α, of $f(x)$ such that $f(\alpha) = 0$.

Traub(1) defines an iteration function, ϕ, as the function which maps the n + 1 approximations $x_i, x_{i-1}, \cdots, x_{i-n}$, to the root α, $f(\alpha) = 0$, into x_{i+1}.

Thus

$$x_{i+1} = \phi(x_i, x_{i-1}, \cdots, x_{i-n}).$$

Iteration function and iteration functions will be abbreviated as I.F.

Traub(1) divides I.F. into four classes, depending on whether they use new information at one or several points, and whether they reuse old information.

If x_{i+1} is determined uniquely by new information at one point, x_i, and no old information is reused, then

$$x_{i+1} = \phi(x_i)$$

and ϕ is defined as a one-point I.F.

If x_{i+1} is determined by new information at one point, x_i, and old information at the n points $x_{i-1}, x_{i-2}, \cdots, x_{i-n}$, then

$$x_{i+1} = \phi(x_i; x_{i-1}, x_{i-2}, \cdots, x_{i-n}) \tag{2.1}$$

and ϕ is defined as a one-point I.F. with memory. The semicolon in (2.1) separates the point at which new data is used from the points at which old data are reused.

Let x_{i+1} be uniquely determined by new information at $x_i, x_{i-1}, \cdots, x_{i-k}$, $k \geq 1$, and no old information reused, then

$$x_{i+1} = \phi(x_i, x_{i-1}, \cdots, x_{i-k})$$

and ϕ is defined as a multipoint I.F.

If x_{i+1} is uniquely determined by new information at the k + 1 points $x_i, x_{i-1}, \cdots, x_{i-k}$, $k \geq 1$, and old information at the n - k points $x_{i-k-1}, \cdots, x_{i-n}$, $n > k$, then

$$x_{i+1} = (x_i, x_{i-1}, \cdots, x_{i-k}; x_{i-k-1}, \cdots, x_{i-n}) \quad (2.2)$$

and ϕ is defined as a multipoint I.F. with memory. The semicolon in (2.2) separates the points at which new data are used from the points at which old data are reused. There are no well-known examples of either multipoint I.F. or multipoint I.F. with memory.

If we have an I.F., ϕ, which generates a sequence of values x_ν

$x_2 = \phi(x_1)$, $x_3 = \phi(x_2)$, $\cdots$, $x_{\nu+1} = \phi(x_\nu)$; and the sequence of points converges to a point α, such that $\alpha = \phi(\alpha)$ and $f(\alpha) = 0$, then Ostrowski(9) defines α as a fixed point of the iteration. Suppose that $\alpha = \phi(\alpha)$. α is called a point of attraction if, for every starting point x_1 within a sufficiently close neighborhood, η, of α, $\alpha - \eta < x_1 < \alpha + \eta$, we have $x_\nu \to \alpha$. α is called a point of repulsion if, for every η neighborhood of α, $\alpha - \eta < x_1 < \alpha + \eta$, the starting value, x_1, fails to generate a sequence x_ν, such that, $x_\nu \to \alpha$.

If $e_{i+1} \equiv \alpha - x_{i+1}$ and $e_i \equiv \alpha - x_i$ and there exists a pair of finite numbers P and C such that $\frac{e_{i+1}}{e_i^P} \to C$, then Traub(1) defines P as the order of the I.F. and C as the asymptotic error constant. A measure of the information used by an I.F., defined as the informational usage d of an I.F. is the number of new pieces of information required per iteration. The notation $\phi \in {}_dI_p$ is used to indicate that ϕ belongs to the class of I.F. of order P and informational usage d. Ostrowski(9) suggests the name "Horner" for the unit of informational usage.

Two well known I.F. are the Newton-Rhapson I.F. and the Regula Falsi I.F. or method of false position, both of which may be generalized to systems of equations.

Scaraborough(10) develops the Newton-Rhapson I.F. in the following manner. Let α represent the roots of $f(x)$ such that $f(\alpha) = 0$. Choose an approximation x_1 to the root α, and let h denote the correction which must be applied to x_1 to give the exact value of the root, so that

$$\alpha = x_1 + h.$$

Then the equation $f(\alpha) = 0$ becomes

$$f(x_1 + h) = 0. \tag{2.3}$$

Expand (2.3) in a Taylor's series. The result is

$$f(x_1 + h) = f(x_1) + hf'(x_1) + \frac{h^2}{2!}f''(x_1 + \xi h),$$

$$0 \le \xi \le 1.$$

Then, since $f(x_1 + h) = 0$,

$$f(x_1) + hf'(x_1) + \frac{h^2}{2!}f''(x_1 + \xi h) = 0.$$

If h is small then the term with h^2 may be neglected and

$$f(x_1) + hf'(x_1) = 0, \tag{2.4}$$

from which,

$$h_1 = -\frac{f(x_1)}{f'(x_1)} .$$

The improved root is

$$x_2 = x_1 + h_1 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

and succeeding approximations are

$$x_3 = x_2 + h_2 = x_2 - \frac{f(x_2)}{f'(x_2)}, \quad x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}, \quad \cdots,$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{2.5}$$

Hildebrand(11) develops the asymptotic error constant for the Newton-Rhapson I.F. Rewrite equation (2.5) in the equivalent form

$$\alpha - x_{n+1} = \alpha - x_n - \frac{f(\alpha) - f(x_n)}{f'(x_n)}. \tag{2.6}$$

Now from the Taylors series expansion

$$f(\alpha) - f(x_n) = (\alpha - x_n)f'(x_n) + \frac{1}{2}(\alpha - x_n)^2 f''(\xi_n) \tag{2.7}$$

where ξ_n lies between x_n and α. If α is a root of the equation such that $f(\alpha) = 0$, then α is a fixed point of the iteration and $\alpha = \phi(\alpha)$. Then, if f" is continuous in the interval between x_n and α, (2.7) becomes

$$(\alpha - x_{n+1}) = (\alpha - x_n) - [(\alpha - x_n)f'(x_n) + \frac{1}{2}(\alpha - x_n)^2 f''(\xi_n)]/f'(x_n),$$

$$\alpha - x_{n+1} = -\frac{1}{2}(\alpha - x_n)^2 \frac{f''(\xi_n)}{f'(x_n)}.$$

Then

$$\frac{\alpha - x_{n+1}}{(\alpha - x_n)^2} \simeq -\frac{1}{2}\frac{f''(\xi_n)}{f'(x_n)}.$$

Thus, if the iteration converges to α, there follows

$$\frac{\alpha - x_{n+1}}{(\alpha - x_n)^2} \simeq -\frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}$$

when n is sufficiently large.

Therefore, by the previous definitions of order and asymptotic error constant, the Newton-Rhapson I.F. is of order two and has an asymptotic error constant approximately equal to $-f''(\alpha)/2f'(\alpha)$. To insure convergence this factor should be smaller than unity in magnitude so that

$$-2 < \frac{f''(\alpha)}{f'(\alpha)} < 2.$$

The Newton-Rhapson I.F. uses new information only at the point x_n, and we must calculate f and f' at each point. Therefore the informational usage is two Horners and the I.F. is a one-point I.F. The Newton-Rhapson I.F. then belongs to the class of I.F. of order 2 and informational usage 2, or $\phi \varepsilon_2 I_2$.

Hildebrand(11) generalizes the Newton-Rhapson I.F. to systems of equations.

For two equations and two unknowns

$$f(x,y) = 0, \; g(x,y) = 0 \; ,$$

expand $f(\alpha,\beta)$ and $g(\alpha,\beta)$ in a Taylor series about x_k, y_k. Then

$$0 = f(\alpha,\beta) = f(x_k,y_k) + (\alpha - x_k)f_x(x_k,y_k) + (\beta - y_k)f_y(x_k,y_k) + \cdots \qquad (2.8)$$

$$0 = g(\alpha,\beta) = g(x_k,y_k) + (\alpha - x_k)g_x(x_k,y_k) + (\beta - y_k(g_y(x_k,y_k) + \cdots \quad (2.9)$$

where the subscripts, x, y, denote the partial derivative of the function with respect to x and y. Replacing (α,β) by (x_{k+1},y_{k+1}) in the right side of (2.8) and (2.9) and neglecting nonlinear terms in $(x_{k+1} - x_k)$, the recurrence formulas are of the form

$$(x_{k+1} - x_k)f_x(x_k,y_k) + (y_{k+1} - y_k)f_y(x_k,y_k) = - f(x_k,y_k) \quad (2.10)$$

$$(x_{k+1} - x_k)g_x(x_k,y_k) + (y_{k+1} - y_k)g_y(x_k,y_k) = - g(x_k,y_k). \quad (2.11)$$

Rather than resolve (2.10) and (2.11) for x_{k+1} and y_{k+1}, it is convenient to solve them as written for $\Delta x_k = x_{k+1} - x_k$ and $\Delta y_k = y_{k+1} - y_k$, which are added to x_k and y_k to yield the subsequent iterates.

When the Jacobian determinant of f and g,

$$J = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix},$$

vanishes at the point (x_k,y_k) the equations (2.10) and (2.11) do not possess a unique solution. More generally, if J vanishes at or near the point (α,β), slow convergence or divergence may be anticipated.

Ostrowski(9) develops the Regula Falsi I.F. in the following way. Let f(x) be defined in some interval I_x. Assume two interpolation points x_1, x_2 in I_x, $x_1 \neq x_2$ and $f(x_1) \neq f(x_2)$.

Approximate f(x) by a linear function which assumes the values $f(x_1)$ and $f(x_2)$,

$$f(x) = \frac{(x - x_1)f(x_2) - (x - x_2)f(x_1)}{x_2 - x_1} . \qquad (2.12)$$

Set the right side of (2.12) equal to zero and solve with respect to x to obtain the next approximation

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} . \qquad (2.13)$$

Equation (2.13) may then be extended in the form

$$x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})} . \qquad (2.14)$$

Let x play the role of x_i and let x_i be x_{i-1}. Then (2.14) may be rewritten as

$$\phi(x) = \frac{x_i f(x) - x f(x_i)}{f(x) - f(x_i)} .$$

Hildebrand(11) develops the order and the asymptotic error constant of the Regula Falsi I.F. in the following manner. It is required that $\phi(\alpha) = \alpha$ and $f(\alpha) = 0$. Define the sequence

$$\gamma_i = \frac{0 - f(x_i)}{\alpha - x_i}$$

so that γ_i will represent the slope of the secant line joining the points $f(\alpha)$ and $f(x_i)$. Rewrite the I.F. as

$$x_{i+1} = x_i - \frac{f(x_i)}{\gamma_i} .$$

Since $f(\alpha) = 0$, then

$$\alpha - x_{i+1} = \alpha - x_i - \frac{f(\alpha) - f(x_i)}{\gamma_i} . \tag{2.15}$$

Then by the mean value theorem, (2.15) becomes

$$\alpha - x_{i+1} = (\alpha - x_i)\left[1 - \frac{f'(\xi_i)}{\gamma_i}\right]$$

and

$$\frac{\alpha - x_{i+1}}{\alpha - x_i} = 1 - \frac{f'(\xi_i)}{\gamma_i} .$$

Thus, if the iteration converges to α, there follows

$$\frac{\alpha - x_{i+1}}{\alpha - x_i} = 1 - \frac{f'(\alpha)}{\gamma_i} .$$

Hence the Regula Falsi I.F. is of order one and has an asymptotic error constant equal to $1 - f'(\alpha)/\gamma_i$. To insure convergence this factor should be smaller than unity in magnitude so that

$$0 < \frac{f'(\alpha)}{\gamma_i} < 2 .$$

The Regula Falsi I.F. (2.14) uses new information at the point x_i and reuses old information at the point x_{i-1}, therefore the I.F. is a one-point I.F. with memory. The I.F. has an informational efficiency of one Horner. The order of the I.F. is one and the informational usage is one, therefore the Regula Falsi I.F. belongs to the class of I.F. of order one and informational efficiency one, or, $\phi\varepsilon_1 I_1$.

Ostrowski(9) generalizes the Regula Falsi I.F. to systems of equations. For two equations and two unknowns

$$f(x,y) = 0 \ , \ g(x,y) = 0 \ ,$$

assume that for the three points (x_1,y_1), (x_2,y_2) and (x_3,y_3) the values of f and g are known. That is, we have

$$\begin{array}{ll} f(x_1,y_1) & g(x_1,y_1) \\ f(x_2,y_2) & g(x_2,y_2) \\ f(x_3,y_3) & g(x_3,y_3) \ . \end{array}$$

Define two linear functions, L_1, L_2, in x and y,

$$L_1 \equiv a_1x + a_2y + a_3, \ L_2 \equiv a_4x + a_5y + a_6$$

by the six conditions

$$\begin{aligned} L_1(x_1,y_1) &= f(x_1,y_1) \ , \ L_2(x_1,y_1) = g(x_1,y_1) \\ L_1(x_2,y_2) &= f(x_2,y_2) \ , \ L_2(x_2,y_2) = g(x_2,y_2) \qquad (2.16) \\ L_1(x_3,y_3) &= f(x_3,y_3) \ , \ L_2(x_3,y_3) = g(x_3,y_3) \ . \end{aligned}$$

Solve with respect to (x,y) the two equations

$$L_1(x,y) = 0 \ , \ L_2(x,y) = 0 \ . \qquad (2.17)$$

To eliminate a_1, a_2, a_3, a_4, a_5, a_6 from the system (2.16) (2.17), first eliminate a_3 and a_6. Subtracting from each equation in (2.17) the corresponding equation from (2.16), two systems of equations are obtained;

$$\begin{aligned} a_1(x - x_1) + a_2(y - y_1) &= - f(x_1,y_1) \\ a_1(x - x_2) + a_2(y - y_2) &= - f(x_2,y_2) \qquad (2.18) \\ a_1(x - x_3) + a_2(y - y_3) &= - f(x_3,y_3) \end{aligned}$$

and

$$\begin{aligned} a_4(x - x_1) + a_5(y - y_1) &= - g(x_1,y_1) \\ a_4(x - x_2) + a_5(y - y_2) &= - g(x_2,y_2) \\ a_4(x - x_3) + a_5(y - y_3) &= - g(x_3,y_3). \end{aligned} \tag{2.19}$$

The result of the elimination of a_1, a_2, a_4, a_5 from (2.18), (2.19) amounts to the statement that the 3×4 matrix

$$\begin{vmatrix} x - x_1 & y - y_1 & f(x_1,y_1) & g(x_1,y_1) \\ x - x_2 & y - y_2 & f(x_2,y_2) & g(x_2,y_2) \\ x - x_3 & y - y_3 & f(x_3,y_3) & g(x_3,y_3) \end{vmatrix}$$

has rank ≤ 2.

The two equations

$$\begin{vmatrix} x - x_1 & x - x_2 & x - x_3 \\ f(x_1,y_1) & f(x_2,y_2) & f(x_3,y_3) \\ g(x_1,y_1) & g(x_2,y_2) & g(x_3,y_3) \end{vmatrix} = 0 \tag{2.20}$$

and

$$\begin{vmatrix} y - y_1 & y - y_2 & y - y_3 \\ f(x_1,y_1) & f(x_2,y_2) & f(x_3,y_3) \\ g(x_1,y_1) & g(x_2,y_2) & g(x_3,y_3) \end{vmatrix} = 0 \tag{2.21}$$

are obtained from the three given approximations, (x_1,y_1), (x_2,y_2), (x_3,y_3), to obtain a fourth improved solution.

If the determinant

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ f(x\ ,y\ ) & f(x\ ,y\ ) & f(x\ ,y\ ) \\ g(x\ ,y\ ) & g(x\ ,y\ ) & g(x\ ,y\ ) \end{vmatrix}$$

is $\neq 0$, then the solution of (2.20) and (2.21) is given by

$$x = \frac{1}{\Delta} \begin{vmatrix} x_1 & x_2 & x_3 \\ f(x_1,y_1) & f(x_2,y_2) & f(x_3,y_3) \\ g(x_1,y_1) & g(x_2,y_2) & g(x_3,y_3) \end{vmatrix} \tag{2.22}$$

and

$$y = \frac{1}{\Delta} \begin{vmatrix} y_1 & y_2 & y_3 \\ f(x_1,y_1) & f(x_2,y_2) & f(x_3,y_3) \\ g(x_1,y_1) & g(x_2,y_2) & g(x_3,y_3) \end{vmatrix} . \tag{2.23}$$

A necessary condition for $\Delta \neq 0$ is that the three points (x_1,y_1), (x_2,y_2), (x_3,y_3) are not collinear. After the fourth approximating point (x_4,y_4) has been found from (2.22) and (2.23) one of the points (x_1,y_1), (x_2,y_2), (x_3,y_3) is dropped and the procedure is repeated, starting from the triplet consisting of the remaining two points and (x_4,y_4).

There are no well known multipoint iteration functions. Traub(1) develops several sets of multipoint I.F. and two of these sets were investigated.

The first set of multipoint I.F. to be investigated is a recursive type of I.F. that can be easily formed on a digital computer. If p is the order of the I.F., the set is defined by

$$\begin{aligned} \phi(x) &= \lambda_p(x) \\ \lambda_j(x) &= \lambda_{j-1}(x) - \frac{f[\lambda_{j-1}(x)]}{f'(x)} \end{aligned} \tag{2.24}$$

for $j = 2, 3, \cdots, p$ and $\lambda_1(x) = x$.

The following two theorems, proved by Traub(1), Chapter 2, are stated as they are used in the development of this set of I.F. The following notation is used; m refers to the multiplicity of the root, α, and $\phi \varepsilon I_p$ denotes that ϕ belongs to the set of I.F. of order p.

Theorem I. Let ϕ be an I.F. such that $\phi^{(p)}$ is continuous in a neighborhood of α. Then ϕ is of order p if and only if

$$\phi(\alpha) = \alpha \ ; \ \phi^{(j)}(\alpha) = 0 \ , \ j = 1, 2, \cdots, p-1 \ ; \ \phi^{(p)}(\alpha) \neq 0.$$

Theorem II. Let m = 1 and $\phi(x) \varepsilon I_p$. Then

$$\left. \frac{d^j f[\phi(x)]}{dx^j} \right|_{x=\alpha} = f'(\alpha)\phi^{(j)}(\alpha) \qquad 0 \leq j < 2p \ .$$

The set is then developed in the following manner.

Let m = 1 and let $\phi(x) \varepsilon I_p$. Let

$$\psi(x) = \phi(x) - \frac{f[\phi(x)]}{f'(x)} \ .$$

Then $\psi(x) \varepsilon I_{p+1}$.

It must then be shown that the conditions of Theorem I are satisfied. If α is a root of f(x) and ϕ is of order p then α is a fixed point of ϕ and $f(\alpha) = 0$, $\phi(\alpha) = \alpha$. Then

$$\psi(\alpha) = \phi(\alpha) - \frac{f(\alpha)}{f'(\alpha)}$$

and

$$\psi(\alpha) = \alpha.$$

The conditions $\psi^{j}(\alpha) = 0$, j = 1, 2, $\cdots$, p, are obtained by induction.

$$\psi'(x) = \phi'(x) - \frac{f'(x)\frac{d}{dx}f[\phi(x)] - f[\phi(x)]f''(x)}{[f'(x)]^2} .$$

Then using Theorem II:

$$\psi'(\alpha) = \phi'(\alpha) - \frac{f'(\alpha)f'(\alpha)\phi'(\alpha) - f(\alpha)f''(\alpha)}{[f'(\alpha)]^2} ;$$

$$\psi'(\alpha) = 0. \tag{2.25}$$

Fix ℓ as any integer such that $\ell \leq p$. Assume that $\psi^{(j)}(\alpha)=0$, $j = 1, 2, \cdots, \ell - 1$. Rewrite the definition of $\psi(x)$ as

$$f'(x)\psi(x) = f'(x)\phi(x) - f[\phi(x)] . \tag{2.26}$$

Differentiating (2.26) ℓ times with respect to x yields

$$\sum_{j=0}^{\ell} C[\ell,j]f^{(\ell-j+1)}(x)\psi^{(j)}(x) = \sum_{j=0}^{\ell} C[\ell,j]f^{(\ell-j+1)}(x)\phi^{(j)}(x) - \frac{d^{\ell}f[\phi(x)]}{dx^{\ell}} \tag{2.28}$$

where $C[\ell,j]$ is a binomial coefficient. Set $x = \alpha$ in (2.28). From the inductive assumption and the hypothesis that $\phi(x)\varepsilon I_p$, it may be concluded that

$$\psi^{(j)}(\alpha) = 0, \ \phi^{(j)}(\alpha) = 0 \qquad j = 1, 2, \cdots, \ell - 1.$$

For $j = \ell$, (2.22) then becomes

$$f'(\alpha)\psi^{(\ell)}(\alpha) = f'(\alpha)\phi^{(\ell)}(\alpha) - f'(\alpha)\phi^{\ell}(\alpha) .$$

Since $m = 1$ implies $f'(\alpha) \neq 0$, it is concluded that $\psi^{(\ell)}(\alpha) = 0$. This completes the induction. Since ℓ is any integer such that $\ell \leq p$, then

$$\psi^{(p+1)}(\alpha) \neq 0 \qquad \psi^{j}(\alpha) = 0 \qquad j = 1, 2, \cdots, p$$

and by Theorem I ψ is of order p+1.

Traub(1) states that the asymptotic error constant for the set of I.F. given by (2.24) may be developed and that it is given by

$$C\phi \rightarrow \frac{1}{2} \left| \frac{f''(\alpha)}{f'(\alpha)} \right|^{p-1} \tag{2.29}$$

where $C\phi$ denotes the asymptotic error constants of the iteration function, ϕ.

The first member of the set defined by (2.24) is the Newton-Rhapson I.F. For each p, p-1 pieces of information are used and no old information is reused, so that the set fits the definition of multipoint I.F. This set of I.F. has the following advantages:

a. Although p-1 evaluations of f are required, only one evaluation of f' is required.

b. Since $1/f'(x)$ need be calculated only once even for high-order I.F., it is well suited for computers having slow division or lacking automatic division. It is also suitable for desk calculators.

c. The recursive definition $\phi(x)$ permits its calculation in a simple loop on a computer.

d. The asymptotic error constant of $\phi(x)$ depends only on $f''(\alpha)/f'(\alpha)$ for p arbitrary.

e. The form of $\phi(x)$ suggests generalization to systems of equations with $1/f'(x)$ replaced by $J^{-1}(x)$ where $J(x)$ is the Jacobian matrix of the system.

The second set of I.F. developed by Traub(1) depend on a number of parameters. The parameters may be chosen to give the I.F. one of the following desirable characteristics:

a. The I.F. is to be of order three or four.

b. The coefficients of the I.F. are to be simple.

c. One or more of the coefficients of the I.F. are to be zero.

d. The asymptotic error constant is not dependent upon derivatives higher than the second.

e. The numerical coefficients in the formula for the asymptotic error constant are to be small.

f. The sampling of f and f' is to be done at reasonable points.

The set of I.F. are defined to be of the form

$$\phi(x) = x - \sum_{k=1}^{p-1} a_k^p \, \omega_k^p \,(x) \qquad (2.30)$$

where

$$\omega_k^p(x) = \frac{f(x)}{f[x + \sum_{j=1}^{k-1} b_{k,j}^p \, \omega_j^p(x)]},$$

$$\omega_1^p(x) = \frac{f(x)}{f'(x)}$$

and $\sum_{j=1}^{0} b_{k,j}^p \, \omega_j(x) = 0$.

In general $\phi(x)$ is a member of the set of I.F. of order p if the parameters a_k^p, $b_{k,j}^p$ are chosen so that $\phi^{(r)}(\alpha) = 0$ for $r = 1, 2, \cdots, p-1$.

Parameters for I.F. of order three and four are developed. By writing (2.30) in a simplified form, the I.F.

$$\phi(x) = x - a_1\omega_1(x) - a_2\omega_2(x) - a_3\omega_3(x) \tag{2.31}$$

is obtained, where

$$\omega_1(x) = \frac{f(x)}{f'(x)} \quad , \quad \omega_2(x) = \frac{f(x)}{f'[x + \beta\omega_1(x)]}$$

and

$$\omega_3(x) = \frac{f(x)}{f'[x + \gamma\omega_1(x) + \gamma\omega_2(x)]} .$$

Take $a_3 = 0$ and define $f(x)/f'(x)$ by $u(x)$. Expand $\omega_2(x)$ into a power series in $u(x)$. Then

$$\omega_2(x) = \frac{f(x)}{f'[x + \beta\omega_1(x)]} = \frac{f(x)}{f'[x + \beta u(x)]} =$$
$$u(x) - 2\,A_2(x)u^2(x) + [4\beta^2A_2^2(x) - 3\beta^2A_3(x)]u^3(x)$$
$$+ \phi[u^4(x)] \ ,$$

where

$$A_j(x) \equiv \frac{f^j(x)}{j!f'(x)} .$$

Hence

$$\begin{aligned}\phi(x) = x - (a_1 + a_2)u(x) + 2\beta a_2A_2(x)u^2(x) \qquad (2.32)\\ - [4\beta^2a_2A_2^2(x) - 3\beta^2a_2A_3(x)]u^3(x) + \phi[u^4(x)] .\end{aligned}$$

Truab(1) proves a theorem which shows that two I.F. of order p can differ only by terms proportional to u^p, where $u = f/f'$. He also develops a I.F., E_s, which is useful in helping to deduce many of the properties of arbitrary I.F.

$$E_s = x + \sum_{j=1}^{s-1} \frac{(-1)^j}{j!} f^j(x) \left|\frac{1}{f'(x)}\frac{d}{dx}\right|^{j-1} \frac{1}{f'(x)} .$$

E_4 is used in the development, and

$$E_4 = x - A_2u^2(x) - (2A_2^3 - A_3)u^3(x).$$

Then subtract E_4 from (2.32) to obtain

$$\phi(x) - E_4 = (1 - a_1 - a_2)u(x) + (1 + 2\beta a_2)A_2(x)u^2(x) + (2 - 4\beta^2 a_2)A_2^2(x)u^3(x) + (-1 + 3\beta^2 a_2)A_3(x)u^3(x) + \phi[u_4(x)] .$$

Then $\phi(x)$ is of order three if

$$a_1 + a_2 = 1 , \; 2\beta a_2 = -1 .$$

The asymptotic error constant is given by

$$\frac{\phi(x) - \alpha}{(x - \alpha)^3} \to 2(\beta + 1)A_2^2(\alpha) - (1 + \frac{3}{2})A_3(\alpha).$$

a_1, a_2 and β are then chosen to give the I.F. the desired characteristics.

For the fourth order case let $a_3 \neq 0$. Expand ω_2 and ω_3 in terms of u(x). The calculations are long and tedious, but the results are

$$\begin{aligned}\phi(x) - E_S(x) &= (1 - a_1 - a_2 - a_3)u(x) + (1 + 2\beta a_2 + 2\nu a_3) \\ &\times A_2(x)u^2(x) + \{2 - 2[2\beta^2 a_2 + 2(\nu^2 + \beta\delta)a_3]\}A_2^2(x)u^3(x) \\ &+ (-1 + 3\beta^2 a_2 + 3\nu^3 a_3)A_3(x)u^3(x) \\ &+ \{5 + 8[\beta^3 a_2 + (\delta\beta^2 + \nu^3 + 2\nu\delta\beta)a_3]\}A_2^3(x)u^4(x) \\ &+ \{-5 - 6[2\beta^3 a_2 + (\delta\beta^2 + 2\nu^3 + 2\nu\delta\beta)a_3]\}A_2(x)A_3(x)u^4(x) \\ &+ [1 + 4(a_2\beta^3 + a_3\nu^3)]A_4(x)u^4(x) + \phi[u^5(x)],\end{aligned}$$

where $\nu = \gamma + \delta$.

Hence $\phi(x)$ is of order 4 if

$$a_1 + a_2 + a_3 = 1$$

$$\beta a_2 + \nu a_3 = -\frac{1}{2}$$

$$\beta^2 a_2 + (\nu^2 + \beta\delta) a_3 = \frac{1}{2}$$

$$\beta^2 a_2 + \qquad \nu^2 a_3 = \frac{1}{3} .$$

Subtracting the 4th equation from the 3rd equation of this system leads to the equivalent system

$$\begin{aligned} a_1 + a_2 + a_3 &= 1 \\ \beta a_2 + \nu a_3 &= -\frac{1}{2} \\ \beta^2 a_2 + \nu^2 a_3 &= \frac{1}{3} \\ \beta\delta a_3 &= \frac{1}{6} . \end{aligned} \tag{2.33}$$

Except for changes of signs and interpretations of the parameters this system is identical with the Runge-Kutta third-order system.

By making $\beta \neq -\frac{2}{3}$, $\nu \neq 0$, $\nu \neq \beta$ the general solution of (2.33) is

$$a_1 = \frac{6\beta\nu + 3(\beta + \nu) + 2}{6\beta\nu} ,$$

$$a_2 = \frac{3\nu + 2}{6\beta(\beta - \nu)} ,$$

$$a_3 = \frac{3\beta + 2}{6\nu(\nu - \beta)} ,$$

$$\gamma = \nu - \frac{\nu(\nu - \beta)}{\beta(3\beta + 2)} ,$$

$$\delta = \frac{\nu(\nu - \beta)}{\beta(3\beta + 2)} .$$

The formula for the asymptotic error constant is given by Traub(1) to be

$$\frac{[\phi(x) - \alpha]}{(x - \alpha)^4} \to [5 + 4\beta(\nu + 1) + \frac{16}{3}\nu] A_2^3(\alpha)$$
$$- (\beta + 1)(5 + 6\nu) A_2(\alpha) A_3(\alpha) + [1 + \frac{4}{3}(\beta + \nu) + 2\beta\nu] A_4(\alpha) .$$

The values of the parameters are chosen to give the I.F. the desired characteristics.

Parameters for this set of I.F. for orders 3 and 4 are tabulated in Table I with their asymptotic error constants.

Ferdinand Freudenstein and Bernard Roth(2) develop the parameter perturbation technique.

To determine a solution of the set of equations

$$f_i(\vec{X}) = 0 \qquad i = 1, 2, \cdots, n \tag{2.34}$$

where $\bar{X}$ is an n-dimensional vector with components x_1, x_2, $\cdots$, x_n and $f_i(\bar{X})$ is of the form

$$f_i(\vec{X}) = \sum_{k=0}^{m_i} P_{ik}\beta_{ik}(\vec{X})$$

where the P_{ik} are parameters (or numbers). First consider a derived set of equations

$$g_i^{(0)}(\vec{X}) = 0\ , \quad i = 1, 2, \cdots, n\ .$$

These derived equations may be any set, with one known root, which belongs to the same family as $f_i(\vec{X})$, that is

$$g_i^{(0)}(\vec{X}) = \sum_{k=0}^{m_1} Q_{ik}^{(0)}\beta_{ik}(\vec{X}).$$

Generally $Q_{ik}^{(0)} \neq P_{ik}$.

The derived equations, $g_i^{(0)}(\vec{X}) = 0$ are "deformed" into the equations $f_i(\vec{X}) = 0$ by means of a finite number, N, of successive small increments in the parameters. Formally define N sets of equation

TABLE I

a_1	a_2	a_3	β	γ	δ	p	$(\phi(x) - \alpha)/(x - \alpha)^p$
0	1	0	-1/2	0	0	3	$A_2^2(\alpha) - (1/4)A_3(\alpha)$*
1/2	1/2	0	-1	0	0	3	$(1/2)A_3(\alpha)$
1/4	3/4	0	-2/3	0	0	3	$(2/3)A_2^2(\alpha)$
5/12	7/12	0	-6/7	0	0	3	$(2/7)[A_2^2(\alpha) + A_3(\alpha)]$
1/6	1/6	4/6	-1	-1/4	-1/4	4	$(1/3)A_2^3(\alpha)$
1/4	0	3/4	-1	-4/9	-2/9	4	$(1/9)[A_2^3(\alpha) + A_4(\alpha)]$
1/4	0	3/4	0	-2/3	-2/3	4	$A_2^3(\alpha) - (2/3)A_2(\alpha)A_3(\alpha) + (1/9)A_4(\alpha)$
1/10	5/10	4/10	-1/3	5/12	-15/12	4	$(1/3)A_2^3(\alpha)$
1/6	4/6	1/6	-1/2	1	-2	4	$-(1/3)A_2^3(\alpha) + (1/2)A_2(\alpha)A_3(\alpha)$
0	1/4	3/4	-1	-1/9	-2/9	4	$(1/9)[5A_2^3(\alpha) - A_4(\alpha)]$

$$*A_j(\alpha) = \frac{f^j(\alpha)}{j!f'(\alpha)}$$

$$g_i^{(j)}(\vec{X}) = \sum_{k=0}^{m_i} Q_{ik}^{(j)} \ \beta_{ik}(\vec{X}) \qquad (2.35)$$

such that

$$g_i^{(N)}(\vec{X}) = f_i(\vec{X}) \text{ and } Q_{ik}^{(j)} = Q_{ik}^{(0)} + (P_{ik} - Q_{ik}^{(0)})j/N. \qquad (2.36)$$

The desired root is obtained by solving the N sets of equations as follows. The known root of $g_i^{(0)}(\bar{X}) = 0$ is used as an initial approximation for the iterative solution of $g_i^{(1)}(\vec{X}) = 0$. Then the root of $g_i^{(1)}(\vec{X}) = 0$ is used as an initial estimate of the root of $g_i^{(2)}(\vec{X}) = 0$, and so forth, until the root of $g_i^{(N)}(\vec{X}) = f_i(\vec{X}) = 0$ is obtained.

So in general start with the root of

$$g_i^{j-1}(\vec{X}) = 0 \qquad i = 1, 2, \cdots, n,$$

as an initial estimate to obtain the root of

$$g_i^{j}(\vec{X}) = 0 \qquad i = 1, 2, \cdots, n,$$

where successive values for the roots, x_i, are determined by the use of some iteration function.

In discussing the convergence of the parameter perturbation algorithm, it is convenient to regard the discrete index j as a continuous variable t. With this substitution, (2.35) can be rewritten as

$$g_i(t,\vec{X}) = \sum_{k=0}^{m_i} Q_{ik}^{(t)} \ \beta_{ik}(\vec{X}) \qquad 0 \le t \le N$$

and, after substituting (2.36), becomes

$$g_i(t,\vec{X}) = \sum_{k=0}^{m_i} (Q_{ik}^{(0)} + (P_{ik} - Q_{ik}^{(0)})\frac{t}{N})\beta_{ik}(\vec{X}).$$

The roots $\vec{X}_f$ of $g_i(t,\vec{X}) = 0$ are also functions of the variable t and we write

$$X_f = X_f(t).$$

A necessary and sufficient condition for the convergence of the algorithm is the convergence of each step. The domain of convergence for a functional iteration is given by a Lipschitz condition with constant less than one. The condition being $\left|\alpha_i - x_i^{k+1}\right| \leq M\left|\alpha_i - x_i^k\right|$ where superscript k denotes the kth step of the functional iteration and $M \leq 1$.

The algorithm converges to a root of $f_i(\vec{X}) = 0$ if the functions $f_i(\vec{X})$ and $g_i^{(0)}(\vec{X})$ are such that

$g_i(t,\vec{X}) = 0$ is continuous for $0 \leq t \leq N$

$X_f(t)$ is continuous for $0 \leq t \leq N$.

Whenever a singularity exists in the domain of convergence the determinant of the Jacobian becomes zero and the method will fail to converge.

Therefore when the algorithm is used on a computer and the determinant of the Jacobian at any step becomes less than some preset value the algorithm may be altered in the following way. The increments in the parameters $Q_{ik}^{(j)}$ are made selectively unequal. The parameters are perturbed one at a time and the determinant of the Jacobian is noted. The variables are then successively incremented to increase the value of the determinant of the Jacobian above some predetermined value. The regular variation may then be re-introduced in a suitably continuous manner.

A second method may also be tried if the determinant of the Jacobian, for the regular variation, falls below some preset value. The method may be rerun with the $g_i^{(j)}(\bar{X})$ defined such that

$$g_i^{(N)} = f_i(\vec{X}) = 0 \text{ and } g_i^{(j)}(\vec{X}) = g_i^{(0)}(\vec{X}) + (f_i(\vec{X}) - g_i^{(0)}(\vec{X}))(\tfrac{j}{N}).$$

III. DISCUSSION

The difficulty of the simultaneous solution of n equations is not just n times the difficulty of solving one equation. For example, the first derivative must be replaced by a matrix of n^2 first partial derivatives; the second derivative by a matrix of n^3 second partial derivatives, and so forth. For this reason the I.F. chosen to be generalized require only function and first derivative evaluations.

The solution, $(\alpha_1, \alpha_2, \cdots, \alpha_n)$, to the system of equations

$$f_i(x_1, x_2, \cdots, x_n) = 0 \;, \; i = 1, 2, \cdots, n \tag{3.1}$$

is to be found. It is assumed that the f_i have as many continuous partial derivatives as are needed in the neighborhood of the root $(\alpha_1, \alpha_2, \cdots, \alpha_n)$. The vector notation $\vec{v} = (v_1, v_2, \cdots, v_n)$ is used. Thus $\vec{X} = (x_1, x_2, \cdots, x_n)$, $\vec{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ and (3.1) becomes

$$F(\vec{X}) = 0 \;. \tag{3.2}$$

Since the subscripts denote components, parenthesized superscripts are used to denote iteration count. Thus the components of the error vector at the kth iteration step are denoted by

$$e_i^{(k)} = x_i^{(k)} - \alpha_i .$$

The I.F. used in the solution of (3.2) are vector-valued functions of a vector value. Thus for multipoint I.F.

$$\vec{X}^{k+1} = \phi[x^{(k)}, x^{(k-1)}, x^{(k-2)}, \cdots, x^{(k-n)}].$$

The definition of order for I.F. applied to systems may be obtained by a generalization of Theorem I. From Theorem I, an I.F. is of order p if

$$\phi(\alpha) = \alpha; \ \phi^{j}(\alpha) = 0 \quad j = 1, 2, \cdots, p-1; \ \phi^{(p)}(\alpha) \neq 0.$$

For vector valued I.F., let $\Phi(\overline{X})$ have p continuous partial derivatives with respect to all components x_{ki}. Then the I.F. is of order p if

$$\Phi(\vec{\alpha}) = \vec{\alpha}$$

$$\frac{\partial^{j}\phi_i(\vec{\alpha})}{\partial x_{k_1}\partial x_{k_2}\cdots\partial x_{k_p}} = 0 \quad \text{for all } 1 \leq j \leq p-1$$

$$1 \leq i, k_1, k_2, \cdots, k_j \leq n;$$

$$\frac{\partial^{p}\phi_i(\vec{\alpha})}{\partial x_{k_1}\partial x_{k_2}\cdots\partial x_{k_p}} \neq 0 \tag{3.3}$$

for at least one value of $i, k_1, k_2, \cdots, k_p$.

Expand $\phi_i(\vec{X})$ in a Taylor series about $\overline{\alpha}$. Then

$$\phi_i(\vec{X}) = \phi_i(\vec{\alpha}) + \sum_{k_1=1}^{n}\left|\frac{\partial\phi_i(\vec{\alpha})}{\partial x_{k_1}}e_{k_1} + \frac{1}{2}\sum_{k_2=1}^{n}\left\{\frac{\partial^2\phi_i(\overline{\alpha})}{\partial x_{k_1}\partial x_{k_2}}e_{k_1}e_{k_2}\right.\right.$$

$$\left.\left. + \frac{1}{6}\left(\sum_{k_3=1}^{n}\frac{\partial^2\phi_i(\overline{\alpha})}{\partial x_{k_1}\partial x_{k_2}\partial x_{k_3}}e_{k_1}e_{k_2}e_{k_3} + \cdots\right)\right\}\right| \tag{3.4}$$

where

$$e_{k_\ell} = x_{k_\ell} - \alpha_{k_\ell}.$$

If $\Phi(\vec{X})\varepsilon I_p$, then from (3.3)

$$\phi_i(\vec{X}) - \vec{\alpha} = \frac{1}{p!} \frac{\partial^p \phi_i(\vec{\alpha})}{\partial x_{k_1} \partial x_{k_2} \cdots \partial x_{k_p}} e_{k_1} e_{k_2} \cdots e_{k_p} . \quad (3.5)$$

The quantity $\frac{1}{p!} \frac{\partial^p \phi_i(\vec{\alpha})}{\partial x_{k_1} \partial x_{k_2} \cdots \partial x_{k_p}}$ is the asymptotic error constant.

The I.F. from (2.24) chosen to be generalized to systems were

$$\phi(x) = x - u(x) = \frac{f[x - u(x)]}{f'(x)} , \; p = 3 \quad (3.6)$$

and

$$\phi(x) = x - u(x) - \frac{f[x - u(x) - f[x - u(x)]/f'(x)]}{f'(x)} ,$$

$$p = 3 . \quad (3.7)$$

The I.F. studied from (2.30) were

$$\phi(x) = x - \frac{1}{4}u(x) - \frac{3}{4} \frac{f(x)}{f'[x - 2/3 \; u(x)]} , \; p = 3 \quad (3.8)$$

and

$$\phi(x) = x - \frac{1}{6}u_x - \frac{1}{6} \frac{f(x)}{f'[x - u(x)]}$$

$$- \frac{4}{6} \frac{f(x)}{f'[x - 1/4 \; u(x) - 1/4 \; [f(x)/f'(x - u(x))]} ,$$

$$p = 4 . \quad (3.9)$$

Each of these I.F. involve $1/f'(z)$ which must be generalized to systems. The Jacobian matrix of $f'(z)$ is defined as the matrix J whose elements are denoted by

$$J_{ij} = \frac{\partial f_i(\vec{z})}{\partial x_i \partial x_j} .$$

If $f'(z)$ is $\neq 0$ then it has an inverse $1/f'(z)$. By analogy, if the determinant of J is nonzero, then J^{-1} exists and may be denoted by H. Then (3.6) generalized to systems would be

$$\phi_i(\vec{X}) = x_i - \sum_{j=1}^{n} \{H_{ij}f_j(\vec{X}) - H_{ij}f_m[\Lambda(\vec{X})]\} \qquad (3.10)$$

where $\Lambda(\vec{X}) = x_i - \sum_{j=1}^{n} H_{ij}f_j(\vec{X})$.

Equation (3.7) generalizes to

$$\phi_i(\vec{X}) = x_i - \sum_{j=1}^{n} \{H_{ij}(\vec{X})f_j(\vec{X}) - H_{ij}(\vec{X})f_j[\Lambda(\vec{X})] - H_{ij}(\vec{X})f_j[\rho(\vec{X})]\}, \qquad (3.11)$$

where $\rho(\vec{X}) = x_i - \sum_{j=1}^{n} H_{ij}[\Lambda(\vec{X})]$, and $\Lambda(\vec{X})$ as in (3.10).

Generalizing (3.8) for systems of equations yields

$$\phi_i(\vec{X}) = x_i - \frac{1}{4}\sum_{j=1}^{n} \{H_{ij}(\vec{X})f_j(\vec{X}) - 3H_{ij}[\beta_j(\vec{X})]f_j(\vec{X})\} \qquad (3.12)$$

where $\beta_j(\vec{X}) = x_j - \frac{2}{3}\sum_{j=1}^{n} H_{ij}(\vec{X})f_j(\vec{X})$.

Equation (3.9), when generalized for systems of equations, becomes

$$\phi_i(\vec{X}) = x_i - \frac{1}{6}\sum_{j=1}^{n} \{H_{ij}(\vec{X})f_j(\vec{X}) - H_{ij}[\beta_j(\vec{X})]f_j(\vec{X}) - 4H_{ij}[\gamma_j(\vec{X})]f_j(\vec{X})\} \qquad (3.14)$$

with

$$\beta_j(\vec{X}) = x_j - \sum_{j=1}^{n} H_{ij}(\vec{X}) f_j(\vec{X})$$

and $\gamma_j(\vec{X}) = x_j - \frac{1}{4} \sum_{j=1}^{n} \{H_{ij}(\vec{X}) f_j(\vec{X}) - H_{ij}[\beta_j(\vec{X})] f_j(\vec{X})\}.$

An estimate of the error in each component at each step of the iteration may be developed for these I.F. Denote the elements of the Jacobian matrix of $f'(\bar{X})$ by J_{ij} and the elements of the inverse matrix by H_{ij}.
Then

$$\sum_{k=1}^{n} H_{ik} J_{kj} = \delta_{ik} \tag{3.15}$$

where δ_{ik} is the Kronecker symbol.

The generalized Newton Rhapson I.F. is

$$\phi_i(\vec{X}) = x_i - \sum_{j=1}^{n} H_{ij}(\vec{X}) f_j(\vec{X}) \ .$$

Let i and k be arbitrary and fixed. Then,

$$\frac{\partial \phi_i(\vec{X})}{\partial x_k} = \partial_{ik} - \sum_{j=1}^{n} \frac{\partial H_{ij}}{\partial x_k}(\vec{X}) f_j(\vec{X}) - \sum_{m=1}^{n} H_{im} J_{mk}(\vec{X}) \ .$$

Set $\vec{X} = \vec{\alpha}$, then

$$\frac{\partial \phi_i(\vec{X})}{\partial x_k} = \delta_{ik} - \sum_{j=1}^{n} \frac{\partial H_{ij}(\vec{\alpha})}{\partial x_k} f_j(\vec{\alpha}) - \delta_{ik}$$

but $f_j(\vec{\alpha}) = 0$, so

$$\frac{\partial \phi_i(\vec{\alpha})}{\partial x_k} = 0 \ . \tag{3.16}$$

Now let i, k, ℓ be arbitrary and fixed. Then,

$$\frac{\partial^2\phi_i(\vec{X})}{\partial x_k\partial x_\ell} = -\sum_{j=1}^{n}\{\frac{\partial^2 H_{ij}(\vec{X})}{\partial x_k\partial x_\ell}f_j(\vec{X}) + \frac{\partial H_{ij}(\vec{X})}{\partial x_k}J_{j\ell}(\vec{X})$$

$$+ \frac{\partial H_{ij}(\vec{X})}{\partial x_\ell}J_{jk}(\vec{X}) + H_{ij}\frac{\partial J_{jk}(\vec{X})}{\partial x_\ell}\}. \qquad (3.17)$$

Differentiating (3.15) with respect to x_q yields

$$\sum_{j=1}^{n} H_{ij}(\vec{X})\frac{\partial J_{jk}(\vec{X})}{\partial x_q} + \sum_{m=1}^{n}\frac{\partial H_{im}(\vec{X})}{\partial x_q}J_{mk}(\vec{X}) = 0 . \qquad (3.18)$$

Setting $\vec{X} = \vec{\alpha}$ in (3.17) and using (3.18) yields

$$\frac{\partial^2\phi_i(\vec{\alpha})}{\partial x_k\partial x_\ell} = -\sum_{j=1}^{n}\{\frac{\partial^2 H_{ij}(\vec{\alpha})}{\partial x_k\partial x_\ell}f_j(\vec{\alpha}) + \frac{\partial H_{ij}(\vec{\alpha})}{\partial x_k}J_{j\ell}(\vec{\alpha})$$

$$- H_{ij}(\vec{\alpha})\frac{\partial J_{jk}(\vec{\alpha})}{\partial x_\ell} + H_{ij}(\vec{\alpha})\frac{\partial J_{jk}(\vec{\alpha})}{\partial x_\ell}\} \qquad (3.19)$$

but $f_j(\vec{\alpha}) = 0$, so that

$$\frac{\partial^2\phi_i(\vec{\alpha})}{\partial x_k\partial x_\ell} = -\sum_{j=1}^{n}\frac{\partial H_{ij}(\vec{\alpha})}{\partial x_k}J_{i\ell}(\vec{\alpha})$$

$$= \sum_{j=1}^{n} H_{ij}(\vec{\alpha})\frac{\partial^2 f_j(\vec{\alpha})}{\partial x_k\partial x_\ell} .$$

Define $Z_{ijk}(\vec{\alpha})$ by

$$Z_{ik\ell} = \sum_{j=1}^{n} H_{ij}(\vec{\alpha})\frac{\partial^2 f_j(\vec{\alpha})}{\partial x_k\partial x_\ell} \qquad (3.20)$$

and substitute into the Taylor series expansion (3.4). Then

$$(x_{i+1} - \alpha) = \frac{1}{2}\sum_{\ell=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} H_{ij}(\vec{\alpha})\frac{\partial^2 f_j(\vec{\alpha})}{\partial x_k\partial x_\ell}(x_k - \alpha_k)(x \;\; - \alpha_\ell).$$

Define $e_i^{(q)}$ as the error in the qth step of the iteration, then

$$e_i^{(q+1)} \simeq \frac{1}{2} \sum_{\ell=1}^{n} \sum_{k=1}^{n} \sum_{j=1}^{n} H_{ij}(\vec{\alpha}) \frac{\partial^2 f_j(\vec{\alpha})}{\partial x_k \partial x_\ell} e_k^{(q)} e_\ell^{(q)} .$$

For the third order case (3.6) define

$$\phi_i(\vec{X}) = \lambda_i(\vec{X}) - \sum_{j=1}^{n} H_{ij}(\vec{X}) f_j[\Lambda(\vec{X})] \tag{3.21}$$

$$\lambda_i(\vec{X}) = x_i - \sum_{k=1}^{n} H_{ik}(\vec{X}) f_k(\vec{X}) .$$

Then $\Lambda(\vec{X})$ is the Newton-Rhapson I.F. From (3.16)

$$\frac{\partial \lambda_i(\vec{\alpha})}{\partial x_k} = 0 \tag{3.22}$$

and

$$\frac{\partial^2 \lambda_i(\vec{\alpha})}{\partial x_k \partial x_\ell} = \sum_{j=1}^{n} H_{ij}(\vec{\alpha}) \frac{\partial^2 f_j(\vec{\alpha})}{\partial x_k \partial x_\ell} . \tag{3.23}$$

Since H_{ij} and J_{ij} are elements of inverse matrices, (3.21) may be rewritten as

$$\sum_{j=1}^{n} \{ J_{ij}(\vec{X}) [\phi_j(\vec{X}) - \lambda_j(\vec{X})] - f_j[\Lambda(\vec{X})] \} = 0 .$$

Let i and k be arbitrary and fixed. Then differentiating (3.21) with respect to x_k yields

$$\sum_{j=1}^{n} \{ \frac{\partial J_{ij}(\vec{X})}{\partial x_k} [\phi_j(\vec{X}) - \lambda_j(\vec{X})] + J_{ij}(\vec{X}) \left| \frac{\partial \phi_j(\vec{X})}{\partial x_k} - \frac{\partial \lambda_j(\vec{X})}{\partial x_k} \right|$$

$$+ \sum_{j=1}^{n} \frac{\partial f_j[\Lambda(\vec{X})]}{\partial \lambda_q(\vec{X})} \frac{\partial \lambda_q(\vec{X})}{\partial x_k} \} = 0 . \tag{3.24}$$

From (3.22) and the fact that $\Lambda(\vec{\alpha}) = \vec{\alpha}$ and $\Phi(\alpha) = \vec{\alpha}$, then

(3.24) becomes

$$\sum_{j=1}^{n} \{ \frac{\partial J_{ij}(\vec{\alpha})}{\partial x_k}[\alpha_j - \alpha_j] + J_{ij}(\vec{\alpha}) \left| \frac{\partial \phi_j(\vec{\alpha})}{\partial x_k} \right| + \sum_{q=1}^{n} \frac{\partial f_j(\vec{\alpha})}{\partial \lambda_q(\vec{\alpha})} \frac{\partial \phi_q(\vec{\alpha})}{\partial x_k} \} = 0$$

or

$$\sum_{j=1}^{n} J_{ij}(\vec{\alpha}) \frac{\partial \phi_j(\vec{\alpha})}{\partial x_k} = 0.$$

Since the Jacobian is assumed nonsingular in a neighborhood of α, then

$$\frac{\partial \phi_j(\vec{\alpha})}{\partial x_k} = 0. \tag{3.25}$$

Differentiate (3.24) with respect to x_ℓ to obtain

$$\sum_{j=1}^{n} \{ \frac{\partial^2 J_{ij}(\vec{X})}{\partial x_k \partial x_\ell}[\phi_j(\vec{X}) - \lambda_j(\vec{X})] + \frac{\partial J_{ij}(\vec{X})}{\partial x_k}\left[\frac{\partial \phi_j(\vec{X})}{\partial x_\ell} - \frac{\partial \phi_j(\vec{X})}{\partial x_\ell}\right]$$
$$+ \frac{\partial J_{ij}(\vec{X})}{\partial x_\ell}\left[\frac{\partial \phi_j(\vec{X})}{\partial x_k} - \frac{\partial \lambda_j(\vec{X})}{\partial x_k}\right] + J_{ij}(\vec{X})\left[\frac{\partial^2 \phi_j(\vec{X})}{\partial x_k \partial x_\ell} - \frac{\partial^2 \lambda_j(\vec{X})}{\partial x_k \partial x_\ell}\right]$$
$$+ \sum_{q=1}^{n}\left[\frac{\partial}{\partial x_\ell} \frac{\partial f_j[\Lambda(\vec{X})]}{\partial \lambda_q(\vec{X})} \frac{\partial \lambda_q(\vec{X})}{\partial x_k} + \frac{\partial f_j[\Lambda(\vec{X})]}{\partial \lambda_q(\vec{X})} \frac{\partial^2 \lambda_q(\vec{X})}{\partial x_k \partial x_\ell}\right] \} = 0. \tag{3.26}$$

Substituting $\vec{X} = \vec{\alpha}$ in (3.26) yields

$$\sum_{j=1}^{n} \frac{\partial^2 J_{ij}(\vec{\alpha})}{\partial x_k \partial x_\ell}[\phi_j(\vec{\alpha}) - \lambda_j(\vec{\alpha})] + \frac{J_{ij}(\vec{\alpha})}{\partial x_k}\left[\frac{\partial \phi_j(\vec{\alpha})}{\partial x_\ell} - \frac{\partial \lambda_j(\vec{\alpha})}{\partial x_\ell}\right]$$
$$+ \frac{J_{ij}(\vec{\alpha})}{\partial x_\ell}\left[\frac{\partial \phi_j(\vec{\alpha})}{\partial x_k} - \frac{\partial \lambda_j(\vec{\alpha})}{\partial x_k}\right] + J_{ij}(\vec{\alpha})\left|\frac{\partial^2 \phi_j(\vec{\alpha})}{\partial x_k \partial x_\ell} - \frac{\partial^2 \phi_j(\vec{\alpha})}{\partial x_k \partial x_\ell}\right]$$
$$+ \sum_{q=1}^{n}\left[\frac{\partial}{\partial x_\ell} \frac{\partial f_j[\Lambda(\vec{\alpha})]}{\partial \lambda_q(\vec{\alpha})} \frac{\partial \lambda_q(\vec{\alpha})}{\partial x_k} + \frac{\partial f_j[\Lambda(\vec{\alpha})]}{\partial \lambda_q(\vec{\alpha})} \frac{\partial^2 \lambda_q(\vec{\alpha})}{\partial x_k \partial x_\ell}\right] \} = 0. \tag{3.27}$$

Using (3.23), (3.25), and $\Phi(\vec{\alpha}) = \vec{\alpha}$, $\Lambda(\vec{\alpha}) = \vec{\alpha}$ and noting that $\lambda_i(\vec{\alpha}) = \alpha_i$ implies

$$\left.\frac{\partial f_i[\Lambda(\vec{X})]}{\partial \lambda_q(\vec{X})}\right|_{\vec{X}=\vec{\alpha}} = J_{iq}(\vec{\alpha}) \ . \tag{3.28}$$

Then (3.27) yields

$$\frac{\partial^2 \phi_j(\vec{\alpha})}{\partial x_k \partial x_\ell} = 0.$$

Hence $\Phi(\vec{X})$ is at least third order.

If (3.27) is differentiated again with respect to x_m, and using (3.23), (3.25), (3.28), $\Phi(\vec{\alpha}) = \vec{\alpha}$ and $\Lambda(\vec{\alpha}) = \vec{\alpha}$, then the following is obtained.

$$\sum_{j=1}^{n} J_{ij}(\vec{\alpha}) \frac{\partial^3 \phi_j(\vec{\alpha})}{\partial x_k \partial x_\ell \partial x_m} = \sum_{j=1}^{n} \left\{ \frac{\partial J_{ij}(\vec{\alpha})}{\partial x_k} \left[\sum_{q=1}^{n} H_{iq}(\vec{\alpha}) \frac{\partial^2 f_q(\vec{\alpha})}{\partial x_m \partial x_\ell} \right] + \frac{\partial J_{ij}(\vec{\alpha})}{\partial x_\ell} \left[\sum_{q=1}^{n} \frac{\partial^2 f_q(\vec{\alpha})}{\partial x_k \partial x_m} \right] + \frac{\partial J_{ij}(\vec{\alpha})}{\partial x_m} \left[\sum_{q=1}^{n} H_{iq} \frac{\partial^2 f_q(\vec{\alpha})}{\partial x_k \partial x_\ell} \right] \right\}. \tag{3.29}$$

If, as before, $Z_{ik\ell}(\vec{\alpha})$ is defined as

$$Z_{ik\ell}(\vec{\alpha}) = \sum_{j=1}^{n} H_{ij}(\vec{\alpha}) \frac{\partial^2 f_j(\vec{\alpha})}{\partial x_k \partial x_\ell}$$

then (3.29) becomes

$$\sum J_{ij}(\vec{\alpha}) \frac{\partial^2 \phi_j(\vec{\alpha})}{\partial x_k \partial x_\ell \partial x_m} = \sum_{q=1}^{n} \left[\frac{\partial J_{iq}(\vec{\alpha})}{\partial x_k} Z_{q\ell m} + \frac{\partial J_{iq}(\vec{\alpha})}{\partial x_\ell} Z_{qkm} + \frac{\partial J_{iq}(\vec{\alpha})}{\partial x_m} Z_{qk\ell} \right]. \tag{3.30}$$

Using H_{ij} and J_{ij} as elements of inverse matrices and permuting the last two subscripts of $Z_{ik\ell}$ leads to

$$\frac{\partial^3 \phi_j(\vec{\alpha})}{\partial x_k \partial x_\ell \partial x_m} = Z_{ikq}(\vec{\alpha}) Z_{q\ell m}(\vec{\alpha}) + Z_{i\ell q}(\vec{\alpha}) Z_{qkm}(\vec{\alpha}) + Z_{imq}(\vec{\alpha}) Z_{qk\ell}(\vec{\alpha}). \tag{3.31}$$

Define $Z_{ik\ell m}(\vec{\alpha}) = \sum_{t=1}^{n} Z_{ikt}(\vec{\alpha}) Z_{t\ell m}(\vec{\alpha})$. Substituting (3.31) into a Taylor series expansion and performing the summation from 1 to n for all j, k, ℓ, m yields

$$e_i^{(q+1)} \approx \frac{1}{2} \sum_{k=1}^{n} \sum_{\ell=1}^{n} \sum_{m=1}^{n} Z_{ik\ell m}(\vec{\alpha}) e_k^{(q)} e_\ell^{(q)} e_m^{(q)}, \tag{3.32}$$

which is an estimate of the error in the ith component at the (q + 1)'st iteration.

The estimate (3.32) is a generalization of the asymptotic error constant, (2.29), for the scalar case.

The generalization to the fourth order case is then

$$e_i^{(q+1)} = \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{\ell=1}^{n} \sum_{m=1}^{n} Z_{ijk\ell m}(\vec{\alpha}) e_j^{(q)} e_k^{(q)} e_\ell^{(q)} e_m^{(q)},$$

where

$$Z_{ijk\ell m}(\vec{\alpha}) = \frac{1}{2} \sum_{t=1}^{n} Z_{ijt}(\vec{\alpha}) Z_{tk\ell m}(\vec{\alpha}).$$

The error estimates of any I.F., whose asymptotic error constant depends on $f''(\alpha)/f'(\alpha)$, may then be formed in the above manner.

IV. NUMERICAL COMPUTATIONS

The iteration functions and parameter perturbation procedure proved satisfactory in the solution of five examples that were examined. The error estimates were developed for utility examples to test the utility of the estimates. The examples were programmed for and computed on the IBM 1620 Model 2 digital computer system.

In calculating the successive approximations to the roots the subroutine GJ1604 was obtained from the Disk-Loaded Library. This subprogram will find the simultaneous solution of a system of nonhomogenous, linear equations using the Gauss-Jordan elimination method.

In order to reduce the round-off errors, the "pivot" for the elimination is chosen as the element of maximum magnitude in the column, and double precision arithmetic or 16 significant digits were retained in the calculation. The tolerances for the successive increments to the x_i in each step were selected to be accurate to 8 decimal places. Each example was run for step sizes two through ten, in the parameter perturbation procedure. The resulting step sizes for which the I.F. gave solutions are tabulated in Table II.

a. Example I.

This problem was chosen to test the development of the error estimate:

$$f_1(\vec{X}) = x_1 - x_2 + x_1 x_2^2 - 1$$

$$f_2(\vec{X}) = x_2 x_1 + x_2 - x_1^2 x_2,$$

where $\vec{\alpha} = (1,0)$.

The Z are calculated in Table III.

Error estimates are given below for each of the I.F. The following information is given for each I.F.:

a. the I.F. and asymptotic error constant in scalar notation,

b. the general error estimate in component notation,

c. the error estimates for the test system.

1a. $\phi(x) = x - u(x) - \dfrac{f[x - u(x)]}{f'(x)}$, $c \simeq \left[\dfrac{f''(\alpha)}{f'(\alpha)}\right]^2$.

b. $e_i^{(q+1)} \simeq \frac{1}{2} Z_{ijk\ell}(\vec{\alpha}) e_j^{(q)} e_k^{(q)} e_\ell^{(q)}$.

c. $e_1^{(q+1)} \simeq 3[e_1^{(q)}]^2 e_2^{(q)} + \frac{9}{2} e_1^{(q)} [e_2^{(q)}]^2 + [e_2^{(q)}]^3$

$e_2^{(q+1)} \simeq 3[e_1^{(q)}]^2 e_2^{(q)} + \frac{3}{2} e_1^{(q)} [e_2^{(q)}]^2 + [e_2^{(q)}]^3$.

2a. $$\phi(x) = x - u(x) - \frac{f[x - u(x)]}{f'(x)} - \frac{f\{x - u(x) - f[x - u(x)]/f'(x)\}}{f'(x)},$$

$c \simeq \dfrac{1}{2}\left|\dfrac{f''(\alpha)}{f'(\alpha)}\right|$.

b. $e_i^{(q+1)} \simeq \frac{1}{2} Z_{ijk\ell m}(\vec{\alpha}) e_j^{(q)} e_k^{(q)} e_\ell^{(q)} e_m^{(q)}$.

c. $$e_1^{(q+1)} \simeq -6[e_1^{(q)}]^3 e_2^{(q)} - 12[e_1^{(q)}]^2 [e_2^{(q)}]^2 - \frac{19}{2} e_1^{(q)} [e_2^{(q)}]^3 - 3[e_2^{(q)}]^4$$

$$e_2^{(q+1)} \simeq -6[e_1^{(q)}]^3 e_2^{(q)} - 6[e_1^{(q)}]^2 [e_2^{(q)}]^2 - \frac{13}{2} e_1^{(q)} [e_2^{(q)}]^3 - [e_2^{(q)}]^4.$$

3a. $$\phi(x) = x - \frac{1}{4}\left\{u(x) + \frac{3f(x)}{f'[x - \frac{2}{3}u(x)]}\right\},$$

$$c \approx \frac{1}{6}\left[\frac{f''(\alpha)}{f'(\alpha)}\right].$$

b. $$e_i^{(q+1)} \approx \frac{1}{6}Z_{ijk\ell}(\vec{\alpha})e_j^{(q)}e_k^{(q)}e_\ell^{(q)}.$$

c. $$e_1^{(q+1)} \approx [e_1^{(q)}]^2e_2^{(q)} + \frac{3}{2}e_1^{(q)}[e_2^{(q)}]^2 + \frac{1}{3}[e_2^{(q)}]^3$$

$$e_2^{(q+1)} \approx 3[e_1^{(q)}]^2e_2^{(q)} + \frac{1}{2}e_1^{(q)}[e_2^{(q)}]^2 + \frac{1}{3}[e_2^{(q)}]^3.$$

4a. $$\phi(x) = x - \frac{1}{6}\left[u(x) + \frac{f(x)}{f'[x - u(x)]} + \frac{4f(x)}{f'\{x - \frac{1}{4}[u(x) + \frac{f(x)}{f'[x - u(x)]}]\}}\right],$$

$$c \approx \frac{1}{24}\left[\frac{f''(\alpha)}{f'(\alpha)}\right]^3.$$

b. $$e_i^{(q+1)} \approx \frac{1}{24}Z_{ijk\ell m}(\vec{\alpha})e_j^{(q)}e_k^{(q)}e_\ell^{(q)}e_m^{(q)}.$$

c. $$e_1^{(q+1)} \approx -\frac{1}{2}[e_1^{(q)}]^3e_2^{(q)} - [e_1^{(q)}]^2[e_2^{(q)}]^2 - \frac{19}{24}e_1^{(q)}[e_2^{(q)}]^3 - 3[e_2^{(q)}]^4$$

$$e_2^{(q+1)} \approx -\frac{1}{2}[e_1^{(q)}]^3e_2^{(q)} - \frac{1}{2}[e_1^{(q)}]^2[e_2^{(q)}]^2 - \frac{13}{24}e_1^{(q)}[e_2^{(q)}]^3 - \frac{1}{12}[e_2^{(q)}]^4.$$

The results for the test function are tabulated in Table VI.

b. Example II.

The system to be solved:

$$f_1(\vec{X}) = x_1^3 - 3x_1x_2^2 - 8$$

$$f_2(\vec{X}) = 3x_1^2x_2 - x_2^3$$

where $\vec{\alpha} = (2,0)$.

The components for the error estimate of this function are tabulated in Table V.

Error estimates are given below for each of the I.F. The following information is given for each I.F.:

a. the I.F. and asymptotic error constant in scalar notation,

b. the general error estimate in component notation,

c. the error estimates for the test function:

1a. $\phi(x) = x - u(x) - \dfrac{f[x - u(x)]}{f'(x)}$, $c \approx \dfrac{1}{2}\left[\dfrac{f''(\alpha)}{f'(\alpha)}\right]$.

b. $e_i^{(q+1)} \approx \frac{1}{2}Z_{ijk\ell}(\vec{\alpha})e_j^{(q)}e_k^{(q)}e_\ell^{(q)}$.

c. $e_1^{(q+1)} \approx \frac{1}{8}[e_1^{(q)}]^3 - \frac{3}{8}e_1^{(q)}[e_2^{(q)}]^2$

$e_2^{(q+1)} \approx \frac{3}{8}[e_1^{(q)}]^2e_2^{(q)} - \frac{1}{8}e_1^{(q)}[e_2^{(q)}]^2 - [e_2^{(q)}]^3$.

2a. $$\phi(x) = x - u(x) - \frac{f[x - u(x)]}{f'(x)} - \frac{f\{x - u(x) - f[x - u(x)]/f'(x)\}}{f'(x)},$$

$c \approx \dfrac{1}{2}\left[\dfrac{f''(\alpha)}{f'(\alpha)}\right]$.

b. $e_i^{(q+1)} \approx \frac{1}{2}Z_{ijk\ell m}(\vec{\alpha})e_j^{(q)}e_k^{(q)}e_\ell^{(q)}e_m^{(q)}$.

c. $e_1^{(q+1)} \approx \frac{1}{16}[e_1^{(q)}]^4 - \frac{3}{8}[e_1^{(q)}]^2[e_2^{(q)}]^2 + \frac{1}{16}[e_2^{(q)}]^4$

$e_2^{(q+1)} \approx \frac{1}{4}[e_1^{(q)}]^3e_2^{(q)} - \frac{1}{4}e_1^{(q)}[e_2^{(q)}]^3$.

$$\text{3a.} \quad \phi(x) = x - \frac{1}{4}\left\{u(x) + \frac{3f(x)}{f'[x - \frac{2}{3}u(x)]}\right\}, \quad c \simeq \frac{1}{6}\left[\frac{f''(\alpha)}{f'(\alpha)}\right]$$

$$\text{b.} \quad e_i^{(q+1)} \simeq \frac{1}{6}Z_{ijk\ell}(\vec{\alpha})\, e_j^{(q)} e_k^{(q)} e_\ell^{(q)}$$

$$\text{c.} \quad e_1^{(q+1)} \simeq \frac{1}{24}[e_1^{(q)}]^3 - \frac{1}{8}e_1^{(q)}[e_2^{(q)}]^2$$

$$e_2^{(q+1)} \simeq \frac{1}{8}[e_1^{(q)}]^2 e_2^{(q)} - \frac{1}{24}e_1^{(q)}[e_2^{(q)}]^2$$

$$\text{4a.} \quad \phi(x) = x - \frac{1}{6}\left[u(x) + \frac{f(x)}{f'[x - u(x)]} + \frac{4f(x)}{f'\left\{x - \frac{1}{4}\left[u(x) + \frac{f(x)}{f'[x - u(x)]}\right]\right\}}\right],$$

$$c \simeq \frac{1}{24}\left[\frac{f''(\alpha)}{f'(\alpha)}\right]$$

$$\text{b.} \quad e_i^{(q+1)} \simeq \frac{1}{24}Z_{ijk\ell m}(\vec{\alpha})\, e_j^{(q)} e_k^{(q)} e_\ell^{(q)} e_m^{(q)}$$

$$\text{c.} \quad e_1^{(q+1)} \simeq \frac{1}{192}[e_1^{(q)}]^3 e_2^{(q)} - \frac{1}{32}[e_2^{(q)}]^2[e_1^{(q)}]^2 + \frac{1}{192}[e_2^{(q)}]^4$$

$$e_2^{(q+1)} \simeq -\frac{1}{48}[e_1^{(q)}]^3 e_2^{(q)} - \frac{1}{48}e_1^{(q)}[e_2^{(q)}]^3$$

The results are given in Table VI.

c. Example III.

The first problem to be tested with the parameter perturbation technique was a third degree system of polynomials

$$f_1(\vec{X}) = P_{10} + P_{11}X_1 + P_{12}X_2^2 + P_{13}X_3^3$$
$$f_2(\vec{X}) = P_{20} + P_{21}X_1 + P_{22}X_2^2 + P_{23}X_3^3$$
$$f_3(\vec{X}) = P_{30} + P_{31}X_1 + P_{32}X_2^2 + P_{33}X_3^3$$

where

$$P = \begin{vmatrix} -18 & 3 & 2 & 1 \\ -16 & 1 & 3 & 1 \\ 6 & -1 & -1 & 1 \end{vmatrix} .$$

The problem was started with the derived set:

$$g_1(\vec{X}) = Q_{10} + Q_{11}X_1 + Q_{12}X_2^2 + Q_{13}X_3^3$$
$$g_2(\vec{X}) = Q_{20} + Q_{21}X_1 + Q_{22}X_2^2 + Q_{23}X_3^3$$
$$g_3(\vec{X}) = Q_{30} + Q_{31}X_1 + Q_{32}X_2^2 + Q_{33}X_3^3$$

where

$$Q = \begin{vmatrix} -3 & 1 & 1 & 1 \\ -3 & 1 & 1 & 1 \\ -3 & 1 & 1 & 1 \end{vmatrix} ,$$

and $x_1 = 1$, $x_2 = 1$, $x_3 = 1$.

Three steps proved sufficient for all the I.F. used and the systems converged tc obtain an exact solution

$$X_1 = 3$$
$$X_2 = 2$$
$$X_3 = 1$$

for the final set. The results are summarized in Table VII where all numbers have been rounded to three decimal places and j refers to the step under consideration.

d. Example IV.

This set is a system of third degree algebraic equations:

$$f_1(\vec{X}) = P_{10}x_1^2 + P_{11}x_2^2 + P_{13}x_1x_2 + P_{14}x_3 + P_{15}$$
$$f_2(\vec{X}) = P_{20}x_1^2 + P_{21}x_2 + P_{23}x_1x_2^2 + P_{24}x_3 + P_{25}$$
$$f_3(\vec{X}) = P_{30}x_1^2 + P_{31}x_2x_3 + P_{33}x_1^2x_2 + P_{34}x_1x_3 + P_{35}$$

where

$$P = \begin{vmatrix} -2 & -5 & 3 & 1 & 13 \\ -8 & -11 & 3 & 1 & 15 \\ 1 & 1 & 5 & 3 & -26 \end{vmatrix} .$$

The parameters of the derived set were

$$Q = \begin{vmatrix} -1 & -3 & 1 & 1 & 10 \\ -5 & -5 & 3 & 1 & 2 \\ 1 & 1 & 2 & 1 & -8 \end{vmatrix}$$

where $x_1 = 1$, $x_2 = 2$, $x_3 = 3$.

All four I.F. gave a solution for step sizes 2 through 10 and all converged to give the exact solution $x_1 = 1$, $x_2 = 2$, $x_3 = 3$. The results of the computations for step size 4 are summarized in Table VIII.

e. Example V.

This problem is another set of polynomial equations

$$f_1(\vec{X}) = P_{10}x_1 + P_{11}x_2 + P_{12}x_2^2 + P_{13}x_2^3 + P_{14}$$

$$f_2(\vec{X}) = P_{20}x_1 + P_{21}x_2 + P_{22}x_2^2 + P_{23}x_2^3 + P_{24}$$

where

$$P = \begin{vmatrix} 1 & -2 & 5 & -1 & -13 \\ 1 & -14 & 1 & 1 & -29 \end{vmatrix} .$$

The derived equations had parameters

$$Q = \begin{vmatrix} 1 & -50 & -13 & -1 & -71 \\ 1 & 106 & 19 & 1 & 129 \end{vmatrix}$$

with a known solution

$$x_1 = 15, \; x_2 = -2.$$

The solutions were found to be exactly 5 and 4 for every case in which the methods converged. The results for step size of six, the only step size for which all reached a solution, are tabulated in Table IX.

f. Example VI.

This example comes from Helm. The system consists of algebraic and transcendental equations of the form

$$f_1(\vec{X}) = P_{10}x_1 + P_{11}x_2 + P_{12}x_3^2 + P_{13}x_4 + P_{14}x_5 + P_{15}x_6 + P_{16}$$

$$f_2(\vec{X}) = P_{20}x_1^2 + P_{21}x_2 + P_{22}x_3 + P_{23}x_4 + P_{24}x_5 + P_{25}x_6^2 + P_{26}$$

$$f_3(\vec{X}) = P_{30}x_1^2 + P_{31}x_2^2 + P_{32}x_3 + P_{33}x_4 + P_{34}x_5 + P_{35}x_6^2 + P_{36}$$

$$f_4(\vec{X}) = P_{40}x_1^2 + P_{41}x_2^3 + \cos(P_{42}x_3) + \sin(P_{43}x_4) + P_{44}x_5 + P_{45}x_6^3 + P_{46}$$

$$f_5(\vec{X}) = e^{P_{50}x_1} + P_{51}x_2 + P_{52}x_3 + P_{53}x_4 + P_{54}x_5 + P_{55}x_6 + P_{56}$$

$$f_6(\vec{X}) = P_{60}x_1^3 + P_{61}x_2 + P_{62}x_3 + P_{63}x_4 + P_{64}x_5 + P_{65}x_6^4 + P_{66}$$

where

$$P = \begin{vmatrix} 2 & -3 & 1 & -2 & 1 & 4 & -1.104 \\ 1 & -2 & 1 & 3 & -3 & 1 & 18.637 \\ 3 & 5 & -6 & -8 & 1 & 6 & -63.714 \\ 2 & -4 & 2 & 3 & 2 & -4 & 121.314 \\ 2 & -9 & 3 & 5 & -6 & 1 & 48.454 \\ -1 & -6 & 4 & 1 & 2 & 1 & 19.340 \end{vmatrix} .$$

The parameters for the derived system were

$$Q = \begin{vmatrix} 2 & -4 & 1 & -3 & 1 & 5 & 3.00 \\ 1 & -2 & 1 & 3 & -4 & 1 & 16.00 \\ 3 & 5 & -1 & -1 & 1 & 8 & -40.00 \\ 2 & -5 & 1 & 1 & 2 & -4 & 127.71 \\ 2 & -1 & 4 & 5 & -8 & 1 & 44.61 \\ -1 & -8 & 4 & 1 & 2 & 1 & 20.00 \end{vmatrix}$$

with a known solution of

$$x_1 = 1$$
$$x_2 = 3$$
$$x_3 = 1$$
$$x_4 = 1$$
$$x_5 = 4$$
$$x_6 = 1 \ .$$

Two steps proved sufficient in all cases and the set of roots accurate to 7 decimal places is found to be

$$x_1 = 1.050267$$
$$x_2 = 3.200070$$
$$x_3 = .05937480$$
$$x_4 = .2019463$$
$$x_5 = 5.002044$$
$$x_6 = 1.000499 \ .$$

The residuals were on the order of 5×10^{-12} in magnitude. The results are tabulated in Table X.

TABLE II

		RECURSIVE I.F.									PARAMETER CHOSEN I.F.								
SAMPLE SIZE		2	3	4	5	6	7	8	9	10	2	3	4	5	6	7	8	9	10
EXAMPLE	ORDER																		
II	3		x	x				x	x			x			x	x	x	x	
	4		x	x				x	x		x	x	x		x	x	x	x	
III	3	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
	4	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
IV	3			x	x	x					x				x			x	x
	4			x		x					x	x			x		x	x	x
V	3	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
	4	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
VI	3	x	x	x				x	x	x	x	x	x	x		x	x	x	x
	4	x	x	x							x	x	x	x	x		x	x	x

TABLE III

$Z_{111} = 0$	$Z_{112} = -2$	$Z_{121} = -1$	$Z_{122} = -2$
$Z_{211} = 0$	$Z_{212} = -2$	$Z_{221} = -1$	$Z_{222} = 0$
$Z_{1111} = 0$	$Z_{1112} = 4$	$Z_{1121} = 2$	$Z_{1122} = 0$
$Z_{1211} = 0$	$Z_{1212} = 6$	$Z_{1221} = 3$	$Z_{1222} = 2$
$Z_{2111} = 0$	$Z_{2112} = 4$	$Z_{2121} = 2$	$Z_{2122} = 0$
$Z_{2211} = 0$	$Z_{2212} = 2$	$Z_{2221} = 1$	$Z_{2222} = 2$
$Z_{11111} = 0$	$Z_{11112} = -8$	$Z_{11121} = -4$	$Z_{11122} = 0$
$Z_{11211} = 0$	$Z_{11212} = -4$	$Z_{11221} = -2$	$Z_{11222} = -4$
$Z_{12111} = 0$	$Z_{12112} = -12$	$Z_{12121} = -6$	$Z_{12122} = 0$
$Z_{12211} = 0$	$Z_{12212} = -10$	$Z_{12221} = -5$	$Z_{12222} = -6$
$Z_{21111} = 0$	$Z_{21112} = -8$	$Z_{21121} = -4$	$Z_{21122} = 0$
$Z_{21211} = 0$	$Z_{21212} = -4$	$Z_{21221} = -2$	$Z_{21222} = -4$
$Z_{22111} = 0$	$Z_{22112} = -4$	$Z_{22121} = -2$	$Z_{22122} = 0$
$Z_{22211} = 0$	$Z_{22212} = -6$	$Z_{22221} = -3$	$Z_{22222} = -2$

TABLE IV

Example I

	q	x_1	e_1	e_1 calculated	x_2	e_2	e_2 calculated
1a	0	1.05	.05		.05	.05	
	1	.9986	-1.73E-04	-2.51E-04	1.15E-04	1.15E-04	2.31E-04
	2	1.0000	5.11E-12	7.06E-12	2.77E-12	2.77E-12	3.49E-11
	3	1.0000	0	4.15E-29	3.47E-27	3.47E-27	6.96E-26
2a	0	1.05	.05		.05	.05	
	1	.9998	-1.73E-04	-2.35E-04	1.57E-04	1.57E-04	1.69E-04
	2	1.0000	5.11E-12	5.86E-12	-3.46E-12	-3.46E-12	4.93E-11
	3	1.0000	0	1.15E-33	3.47E-26	3.47E-26	3.56E-26
3a	0	1.05	.05		.05	.05	
	1	.9987	-1.26E-04	-3.54E-04	4.98E-05	4.98E-05	6.05E-05
	2	1.0000	5.37E-13	6.06E-12	1.18E-13	5.37E-13	4.20E-11
	3	1.0000	0	3.40E-31	1.00E-28	1.00E-28	2.33E-31
4a	0	1.05	.05		.05	.05	
	1	.9989	-2.08E-05	-3.05E-05	2.68E-05	2.68E-05	3.01E-05
	2	1.000	6.22E-20	6.35E-19	2.29E-15	2.29E-15	4.32E-15
	3	1.000	0	1.73E-77	6.81E-31	6.81E-31	1.18E-77

TABLE V

$Z_{111} = 1/2$	$Z_{112} = 0$	$Z_{121} = 0$	$Z_{122} = -1/2$
$Z_{211} = 0$	$Z_{212} = 1/2$	$Z_{221} = 1/2$	$Z_{222} = 0$
$Z_{1111} = 1/4$	$Z_{1112} = 0$	$Z_{1121} = 0$	$Z_{1122} = -1/4$
$Z_{1211} = 0$	$Z_{1212} = -1/4$	$Z_{1221} = -1/4$	$Z_{1222} = 0$
$Z_{2111} = 0$	$Z_{2112} = 1/4$	$Z_{2121} = 1/4$	$Z_{2122} = 0$
$Z_{2211} = 1/4$	$Z_{2212} = 0$	$Z_{2221} = 0$	$Z_{2222} = -1/4$
$Z_{11111} = 1/8$	$Z_{11112} = 0$	$Z_{11121} = 0$	$Z_{11122} = -1/8$
$Z_{11211} = 0$	$Z_{11212} =-1/8$	$Z_{11221} = -1/8$	$Z_{11222} = 0$
$Z_{12111} = 0$	$Z_{12112} = -1/8$	$Z_{12121} = -1/8$	$Z_{12122} = 0$
$Z_{12211} = -1/8$	$Z_{12212} = 0$	$Z_{12221} = 0$	$Z_{12222} = 1/8$
$Z_{21111} = 0$	$Z_{21112} = 1/8$	$Z_{21121} = 1/8$	$Z_{21122} = 0$
$Z_{21211} = 1/8$	$Z_{21212} = 0$	$Z_{21221} = 0$	$Z_{21222} = -1/8$
$Z_{22111} = 1/8$	$Z_{22112} = 0$	$Z_{22121} = 0$	$Z_{22122} = -1/8$
$Z_{22211} = 0$	$Z_{22212} = -1/8$	$Z_{22221} = -1/8$	$Z_{22222} = 0$

TABLE VI

Example III

	q	x_1	e_1	e_1 calculated	x_2	e_2	e_2 calculated
1a	0	2.02	.02		.03	.03	
	1	2.0000	1.27E-05	5.75E-06	-3.56E-06	3.56E-06	1.12E-06
	2	2.0000	2.E-15	2.10E-17	8.51E-16	-8.51E-16	-1.377E-17
	3	2.0000	1.E-15	3.32E-52	-8.51E-31	8.51E-31	1.96E-51
2a	0	2.02	.02		.03	.03	
	1	2.0000	3.41E-07	7.43E-08	-4.39E-07	-4.39E-07	-7.5E-08
	2	2.0000	0	-7.77E-30	1.18E-26	1.18E-26	-1.30E-31
	3	2.0000	0	0	0	0	0
3a	0	2.02	.02		.03	.03	
	1	1.9999	-1.5E-06	-1.9E-06	-5.97E-04	-5.97E-04	-1.47E-05
	2	2.0000	1.1E-13	2.29E-19	-8.29E-10	-8.29E-10	6.78E-11
	3	2.0000	0	-1.27E-56	9.1E-23	9.1E-23	6.9E-24
4a	0	2.02	.02		.03	.03	
	1	1.999	-5.57E-05	-6.19E-05	-6.32E-04	-6.32E-04	-6.98E-04
	2	1.999	-6.07E-08	-3.12E-07	-4.05E-08	-4.05E-08	-6.25E-09
	3	2.000	2.26E-15	-3.5E-15	-2.25E-15	-2.25E-15	-3.E-15

TABLE VII

Example III

j	Q_{10}	Q_{11}	Q_{12}	Q_{13}	Q_{20}	Q_{21}	Q_{22}	Q_{23}	Q_{30}	Q_{31}	Q_{32}	Q_{33}	x_1	x_2	x_3
0	-3.00	1.000	1.000	1	-3.000	1	1.000	1	-3.00	1.000	1.000	1	1.000	1.000	1.000
1	-8.00	1.667	1.333	1	-7.333	1	1.667	1	0.000	.333	.333	1	3.000	2.000	-1.326
2	-13.00	2.333	1.667	1	-11.67	1	2.333	1	3.000	-.333	-.333	1	3.000	2.000	-.8736
3	-16.00	3.000	2.000	1	-16.00	1	3.000	1	6.000	-1.000	-1.000	1	3.000	2.000	1.000

TABLE VIII

Example IV

j	Q_{10}	Q_{11}	Q_{12}	Q_{13}	Q_{14}	Q_{20}	Q_{21}	Q_{22}	Q_{23}	Q_{24}	Q_{30}	Q_{31}	Q_{32}	Q_{33}	Q_{34}	x_1	x_2	x_3
0	-1.0	-3.0	1.0	1	10.00	-5.0	-5.0	3	1	2.0	1	1	2.0	1.0	-8.0	1.00000	2.00000	1.00000
1	-1.2	-3.4	1.4	1	10.6	-5.6	-6.2	3	1	4.6	1	1	2.6	1.4	-11.6	1.04915	1.99951	1.37732
2	-1.4	-3.8	1.8	1	11.2	-6.2	-7.4	3	1	7.2	1	1	3.2	1.8	-15.2	1.04809	2.00372	1.18143
3	-1.6	-4.2	2.2	1	11.8	-6.8	-8.6	3	1	9.8	1	1	3.8	2.2	-18.8	1.03366	2.00545	2.240730
4	-1.8	-4.6	2.6	1	12.4	-7.4	-9.8	3	1	12.4	1	1	4.4	2.6	-22.4	1.01637	2.00404	2.63808
5	-2.0	-5.0	3.0	1	13.0	-8.0	-11.0	3	1	15.0	1	1	5.0	3.0	-26.0	1.00000	2.00000	3.00000

TABLE IX

Example V

j	Q_{10}	Q_{11}	Q_{12}	Q_{13}	Q_{14}	Q_{20}	Q_{21}	Q_{22}	Q_{23}	Q_{24}	x_1	x_2
0	1	-5.0	-13	-1	-71.00	1	106	19	1	129.00	15.00000	-2.00000
1	1	-4.2	-10	-1	-61.33	1	86	16	1	102.67	10.49967	-1.91862
2	1	-3.4	-7	-1	-51.67	1	66	13	1	76.33	6.87003	-1.82406
3	1	-2.6	-4	-1	-42.00	1	46	10	1	50.00	4.32878	-1.70561
4	1	-1.8	-1	-1	-32.33	1	26	7	1	23.67	3.39282	-1.53727
5	1	-1.0	2	-1	-22.67	1	6	4	1	-2.67	5.84272	-1.21126
6	1	-2.0	5	-1	-13.00	1	-14	1	1	-29.00	5.00000	4.00000

TABLE X

Example VI

j	$g_1(\overline{X})$	$g_2(\overline{X})$	$g_3(\overline{X})$	$g_4(\overline{X})$	$g_5(\overline{X})$	$g_6(\overline{X})$	Roots
0	$Q_{10}=2$	$Q_{20}=1$	$Q_{30}=3$	$Q_{40}=2$	$Q_{50}=2$	$Q_{60}=-10$	$x_1=1.000000$
	$Q_{11}=-4$	$Q_{21}=-2$	$Q_{31}=5$	$Q_{41}=-5$	$Q_{51}=-10$	$Q_{61}=-8$	$x_2=3.000000$
	$Q_{12}=1$	$Q_{22}=1$	$Q_{32}=-10$	$Q_{42}=1$	$Q_{52}=4$	$Q_{62}=4$	$x_3=1.000000$
	$Q_{13}=-3$	$Q_{23}=3$	$Q_{33}=-10$	$Q_{43}=1$	$Q_{53}=5$	$Q_{63}=1$	$x_4=1.000000$
	$Q_{14}=1$	$Q_{24}-4$	$Q_{34}=1$	$Q_{44}=2$	$Q_{54}=-8$	$Q_{64}=2$	$x_5=4.000000$
	$Q_{15}=5$	$Q_{25}=1$	$Q_{35}=8$	$Q_{45}=-4$	$Q_{55}=1$	$Q_{65}=1$	$x_6=1.000000$
	$Q_{16}=3$	$Q_{26}=16$	$Q_{36}=-40$	$Q_{46}=127.71$	$Q_{56}=44.61$	$Q_{66}=20$	
1	$Q_{10}=2$	$Q_{20}=1$	$Q_{30}=3$	$Q_{40}=2$	$Q_{50}=2$	$Q_{60}=-10$	$x_1=1.040256$
	$Q_{11}=-3.5$	$Q_{21}=-2$	$Q_{31}=5$	$Q_{41}=-4.5$	$Q_{51}=-9.5$	$Q_{61}=-7$	$x_2=3.093460$
	$Q_{12}=1$	$Q_{22}=1$	$Q_{32}=-8$	$Q_{42}=1.5$	$Q_{52}=3.5$	$Q_{62}=4$	$x_3=.630643$
	$Q_{13}=-2.5$	$Q_{23}=3$	$Q_{33}=-9$	$Q_{43}=2$	$Q_{53}=5$	$Q_{63}=1$	$x_4=.656787$
	$Q_{14}=1$	$Q_{24}=-3.5$	$Q_{34}=1$	$Q_{44}=2$	$Q_{54}=-7$	$Q_{64}=2$	$x_5=4.521242$
	$Q_{15}=4.5$	$Q_{25}=1$	$Q_{35}=7$	$Q_{45}=-4$	$Q_{55}=1$	$Q_{65}=1$	$x_6=1.004803$
	$Q_{16}=.948$	$Q_{26}=17.319$	$Q_{36}=-51857$	$Q_{46}=124.512$	$Q_{56}=46.532$	$Q_{66}=19.670$	

TABLE X

(continued)

j	$g_1(\bar{X})$	$g_2(\bar{X})$	$g_3(\bar{X})$	$g_4(\bar{X})$	$g_5(\bar{X})$	$g_6(\bar{X})$	Roots
2	$Q_{10}=2$	$Q_{20}=1$	$Q_{30}=3$	$Q_{40}=2$	$Q_{50}=2$	$Q_{60}=-10$	$x_1=1.050267$
	$Q_{11}=-3$	$Q_{21}=-2$	$Q_{31}=5$	$Q_{41}=-4$	$Q_{51}=-9$	$Q_{61}=-6$	$x_2=3.200070$
	$Q_{12}=1$	$Q_{22}=1$	$Q_{32}=-6$	$Q_{42}=2$	$Q_{52}=3$	$Q_{62}=4$	$x_3=.0593748$
	$Q_{13}=-2$	$Q_{23}=3$	$Q_{33}=-8$	$Q_{43}=3$	$Q_{53}=5$	$Q_{63}=1$	$x_4=.2019463$
	$Q_{14}=1$	$Q_{24}=-3$	$Q_{34}=1$	$Q_{44}=2$	$Q_{54}=-6$	$Q_{64}=2$	$x_5=5.002044$
	$Q_{15}=4$	$Q_{25}=1$	$Q_{35}=6$	$Q_{45}=-4$	$Q_{55}=1$	$Q_{65}=1$	$x_6=1.000499$
	$Q_{16}=1.104$	$Q_{26}=18.637$	$Q_{36}=-63.714$	$Q_{46}=121.314$	$Q_{56}=48.454$	$Q_{66}=19.34$	

IV. CONCLUSIONS

The multipoint iteration functions used were found to be very successful when applied to systems in connection with the parameter perturbation procedure. The iteration functions, in which the parameters are chosen, have an asymptotic error constant smaller than the corresponding iteration functions which are recursively formed. Those iteration functions with the smaller asymptotic error constant should converge in more cases than those with a larger asymptotic error constant, and the results show this to be true.

The estimates for the error at each step of the iteration were found to be very good. However, since the knowledge of the estimate presupposes the knowledge of the root, the methods are not very useful. The methods may be used only to test the iteration functions one may develop. The amount of hand calculation needed to develop the components also limit their use even as a test.

The convergence of the methods to a solution of the unknown system was found to be highly dependent on the number of steps N. The number of steps needed depends on the parameters of the system of equations to be solved and the parameters of the derived system. If the parameters differ by a significant amount the number of steps probably should be chosen quite large. If the methods do not reach a solution for several step sizes then one must revert to one of the modifications of the parameter perturbation procedure listed in Chapter II.

The parameter perturbation procedure appears to eliminate the problems of choosing a good initial approximation to a desired root.

The author recommends the use of the parameter perturbation procedure with one of the iteration functions, in which the parameters are chosen, for small systems and one of the recursively formed iteration functions for large systems or in systems where the Jacobian matrix involves the evaluation of transcendental functions.

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